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# EXISTENCE AND UNIQUENESS OF PACKINGS WITH SPECIFIED COMBINATORICS

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ABSTRACT

Generalizations of the Andreev–Thurston circle packing theorem are proved. One such result is the following.

*Let $G = G(V)$ be a planar graph, and for each vertex $v \in V$, let $\mathfrak{F}_v$ be a proper 3-manifold of smooth topological disks in $S^2$, with the property that the pattern of intersection of any two sets $A, B \in \mathfrak{F}_v$ is topologically the pattern of intersection of two circles (i.e., there is a homeomorphism $h : S^2 \to S^2$ taking $A$ and $B$ to circles). Then there is a packing $P = (P_v : v \in V)$ whose nerve is $G$, and which satisfies $P_v \in \mathfrak{F}_v$ for $v \in V$. ('The nerve is $G$' means that two sets, $P_v, P_u$, touch, if, and only if, $u \leftrightarrow v$ is an edge in $G$.)*

In the case where $G$ is the 1-skeleton of a triangulation, we also give a precise uniqueness statement. Various examples and applications are discussed.

## 1. Introduction

Consider a packing of circles in the sphere or the plane; that is, an indexed collection of circles $Q = (Q_v : v \in V)$ having disjoint interiors. The nerve of the packing is a graph whose vertex set is $V$ and which describes the combinatorics of the packing. An edge will appear between two vertices in $V$ if, and only if, the corresponding circles touch. One sees immediately that the nerve of such a circle packing is planar. It was an observation of Thurston ([Th1], [Th2]) that Andreev's theorem ([An1], [An2]) implies that the converse also holds. Given a finite planar graph (without loops or multiple edges) there exists a circle packing on the sphere whose nerve is the given graph. Moreover, when the graph is a triangulation (i.e., the 1-skeleton of a triangulation of the sphere) this circle packing is unique up to Möbius transformations. We will refer to this beautiful fact as the Andreev–

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Thurston circle packing theorem. It has interesting applications, including applications to the theory of conformal mappings. See [Th1], [R–S], [He], [Sch1].

In [Sch1] we have generalized the Andreev–Thurston circle packing theorem to packings of sets other than circles. (The nerve of these more general packings is defined in the same manner, and is still planar in the well-behaved cases.) One such generalization is the following.

1.1. Convex Packing Theorem. *Let $G$ be a finite planar graph on the vertex set $V$. For each $v \in V$ let $P_v$ be some smooth convex body in the plane. Then there exists a packing $Q = (Q_v : v \in V)$ in the plane whose nerve is $G$, and with $Q_v$ positively homothetic to $P_v$ for every $v \in V$.*

One obtains the existence part of the Andreev–Thurston theorem from the above when one takes all the $P_v$ to be circles. The packing can be lifted to the sphere via stereographic projection.

In the metric packing theorem, which also appears in [Sch1], for each $v \in V$ one is given a Riemannian metric $d_v$ on the sphere, and one gets a packing $Q$ with the specified nerve and where each $Q_v$ is a ball for the metric $d_v$.

A shortcoming of these results in [Sch1] is that they do not contain any uniqueness statement, as does the Andreev–Thurston theorem. In fact, the arguments there also give packings for which no reasonable uniqueness statement holds. In this note we prove a packing theorem (see 3.2, 3.5) which includes the convex packing theorem and the metric packing theorem as particular cases, and has the virtue of having a precise uniqueness statement. It is just about the most general packing theorem of this kind that one could hope for (it is more general than I have ever hoped for). It has applications to conformal uniformizations of multiply connected domains (as in [Sch1]). Other applications, as well as precise statements of the results, are presented in Section 3, after we introduce some definitions in Section 2.

The technique here is independent of that in [Sch1]. This paper is self-contained, except for a use of the incompatibility theorem of [Sch2], which is actually our central tool.

## 2. Some definitions and notations

Some of the definitions we use appear also in [Sch2]; we repeat them for the sake of completeness.

For us, a *packing* will mean an indexed collection $P = (P_v : v \in V)$ of compact connected sets in the sphere $S^2$ or in the plane, with the property that the interior of each set $P_v$ is disjoint from the other sets $P_w$, $w \neq v$. A packing is *nondegenerate* if no point belongs to more than two of the sets in the packing. When we use the term 'packing', we will mean 'nondegenerate packing', unless we specifically state otherwise.

It is tempting to consider only packings by smooth topological disks, for the sake of simplicity. There are two reasons for not doing so. First, a nice application is for packings of balls of path metrics, and these are sometimes not topological disks (and therefore not smooth), even in the Riemannian case. Second, dealing with the more general case does not involve any additional significant complications, mostly a few more definitions. We will work mostly with sets which we call blunt and disklike (definitions follow). The reader may wish to consider only the case where disklike means topological disk, and blunt means smooth.

For any $A \subset S^2$ let $A^c = S^2 - A$ denote the complement of $A$.

A proper subset $A \subset S^2$ will be called *disklike* if it is the closure of its interior, its interior is connected, and the complement of any connected component of $A^c$ is a topological disk. A disklike $A$ is *blunt* if for every point $p \in \partial A$ there is a smooth set contained in $A$ and containing $p$. (To put it differently, the angle of $A$ at any boundary point of $A$ is at least $\pi$, but we use the above definition, because we do not want to worry if the angle is defined.) The idea is that a packing of blunt sets has to be nondegenerate. For example, a ball of a Riemannian (or Finsler) metric on $S^2$ is disklike and blunt.

The *nerve* of a packing $P = (P_v : v \in V)$ is a graph whose vertex set is $V$, and which is defined by the property that there is an edge between two distinct vertices, $v, w$, if and only if the corresponding sets, $P_v, P_w$, intersect. Note that there is at most a single edge between $v, w$ in the nerve, even if $P_v$ and $P_w$ intersect in more than one place. If $G$ is a graph, we will use the notation $G(V)$ to mean that $V$ is the vertex set of $G$.

Let $T$ be a triangulation of the sphere $S^2$, and let $G$ be the 1-skeleton of $T$, considered as an abstract graph. It is easy to see that $G$ determines $T$ up to a homeomorphism of $S^2$. For this reason we will be sloppy, and will not distinguish between a triangulation and its 1-skeleton.

Let $G$ be a finite graph embedded in $S^2$, and let $B$ be a simple closed path in $G$. We will say that $G$ is a triangulation with boundary $B$, if $G - B$ is connected, and if $G$ has an embedding in the sphere where all the connected components of the complement are triangles (that is, bounded by three edges of $G$), except possibly for one component bounded by $B$. If $B$ has precisely four vertices, then $G$ will be called a *triangulation of a quadrilateral*.

A *quadrilateral* is a closed topological disk $D$ in $S^2$ with four distinguished points $p_0,p_1,p_2,p_3$ on its boundary that are oriented clockwise with respect to the interior of $D$. $D_i$ will be used to denote the arc of the boundary of this quadrilateral which extends clockwise from $p_{i-1}$ to $p_i$, with $p_4$ standing for $p_0$. Such a quadrilateral will be denoted by $(D_1,D_2,D_3,D_4)$, by $D(p_0,p_1,p_2,p_3)$, or just by $D$, depending on convenience.

Similarly, a trilateral $D=(D_1,D_2,D_3)$ is defined as a topological disk with three distinct distinguished clockwise oriented points on its boundary. These are referred to as its vertices. A trilateral $D$ will be termed *cornered* if there is no smooth set contained in it and containing one of its vertices (e.g., the angles there are $<\pi$). A trilateral is *decent* if it is cornered and the intersection of $\partial D$ with any two smooth interiorwise disjoint subsets of $D$ is empty (i.e., it has no inward cusps of angle $2\pi$).

The reason for this definition is that we will consider packings $(P_v : v\in V)$ in a trilateral $D=(D_1,D_2,D_3)$ which have $D_1,D_2,D_3$ as three of the packed sets, say $P_a=D_1$, $P_b=D_2$, $P_c=D_3$. If the other sets in the packing are blunt, and if the trilateral is decent, then the packing is automatically nondegenerate.

We use the following notion of convergence of subsets of $S^2$. Let $A_n$, $n=1,2,3,\ldots$, be a sequence of closed subsets of $S^2$. We shall say that this sequence converges to the set $A\subset S^2$ if $\limsup A_n=\liminf A_n=A$, *and* $A^c=$ interior($\limsup A_n^c$). ($\limsup A_n$ is the set of all accumulation points for sequences $(x_n)$ with $x_n\in A_n$. $\liminf A_n$ is the set of all limit points for such converging sequences $(x_n)$.) The reason for this definition is the following. Let $(A_n),(B_n)$ be sequences of closed subsets of $S^2$. Assume that each $A_n$ is the closure of its interior, that interior$(A_n)\cap B_n=\emptyset$, and that $A_n\to A$, $B_n\to B$. Then it follows that interior$(A)\cap B=\emptyset$. This property will allow us to take limits of packings and obtain a packing.

Let $U$ be an open subset of $S^2$, and let $\mathfrak{F}$ be some collection of subsets of $U$. $\mathfrak{F}$ will be called a *continuous* collection on $U$ if for every sequence of sets $(A_n)$ in $\mathfrak{F}$ which are all contained in some compact proper subset of $U$ there is some subsequence $(A_{n_i})$ so that $A_{n_i}$ converges to some set $A\in\mathfrak{F}$, or $\limsup A_{n_i}=$ (a point).

## 3. The main results and some applications

To prepare for the statement of the main theorem let us consider an elementary packing problem. Let $D=(D_a,D_b,D_c)$ be a trilateral in the sphere or in the plane. We will say that a set $A$ *packs* this trilateral $D$ if $A\subset D$ and $A$ intersects each of the three edges of $D, D_a,D_b,D_c$.

It is not hard to see, and is also proved below, that the collection of all circles on the sphere has the property that for each cornered trilateral in $S^2$ there is a unique circle which packs it. The same statement holds in the plane for the collection of all sets positively homothetic to a given smooth strictly convex set, or for the collection of balls of a given Riemannian or Finsler metric. (See 3.3 below.) It turns out that this is the essential feature which permits us to 'pack' these collections.

3.1. Definition. Let $U$ be some open subset of $S^2$, and let $\mathfrak{F}$ be a continuous collection of blunt disklikes in $U$. If for every cornered trilateral in $U$ there is a unique set in $\mathfrak{F}$ which packs that trilateral, then we will say that $\mathfrak{F}$ is a *packable* collection on $U$.

Our main result justifies this definition:

3.2. Packing Theorem. *Let $U$ be some open subset of $S^2$. Let $T = T(V)$ be a triangulation of the sphere $S^2$, let $[a,b,c]$ be a triangle in $T$, and for $v \in V - \{a,b,c\}$ let $\mathfrak{F}_v$ be a packable collection on $U$. Then, given a decent trilateral $D = (D_a, D_b, D_c) \subset U$, there exists a unique packing $P = (P_v : v \in V)$ contained in $D$ whose nerve is $T$ and which satisfies $P_v \in \mathfrak{F}_v$, $v \in V - \{a,b,c\}$ and $P_v = D_v$, $v = a,b,c$.*

Note that there are no requirements on any relation between the collections $\mathfrak{F}_v$, but only requirements on each particular collection. This is quite surprising.

The definition of packable, 3.1, was chosen, from other equivalent definitions, because it is most natural, but not because it is easy to verify. Below (3.5), we will see that there are equivalent, more concrete, definitions.

A particular case of Theorem 3.2 says that given a triangulation $T(V)$ of the sphere, and for each $v \in V$ a packable collection $\mathfrak{F}_v$ on $S^2$, there exists a packing $P = (P_v : v \in V)$ whose nerve is $T$ and which satisfies $P_v \in \mathfrak{F}_v$ for $v \in V$. Furthermore, this packing is unique once the sets corresponding to three vertices of a triangle are fixed.

If $G = G(V)$ is a planar graph, but not a triangulation, and for $v \in V$, $\mathfrak{F}_v$ is packable on $S^2$, then we can certainly embed $G$ in a triangulation of the sphere which does not have any edges between vertices in $G$, except for those edges which are already in $G$. Thus, using Theorem 3.2, we see that there exists a packing $P = (P_v : v \in V) \subset S^2$ whose nerve is $G$ and which satisfies $P_v \in \mathfrak{F}_v$, $v \in V$. So the existence part of Theorem 3.2 also applies to planar graphs which are not triangulations.

Our method of proving 3.2 is the following. First, the incompatibility theorem of [Sch2] is used to establish uniqueness. Uniqueness then gives the continuity of the packings in the data. Continuity is then used with induction to establish existence.

One application of 3.2 is given by the following.

3.3. Metric Packing Theorem. *Let $U \subset S^2$ be open and connected, and let $d$ be a Riemannian [or, more generally, a possibly nonsymmetric Finsler] metric*† *on $U$. Then the collection of all (proper) balls of $d$ is packable on $U$.*

By a 'proper' ball, we mean a compact ball of positive radius which is strictly contained in $U$.

When one works with nonsymmetric metrics there are two distinct kinds of balls, left balls, $\{x : d(c,x) \le r\}$, and right balls, $\{x : d(x,c) \le r\}$. The above holds when one considers as 'balls' either kind (but, for uniqueness, not both). We will restrict our attention to left balls.

The following important special case of the metric packing theorem uses the more general Finsler metrics. Let $C$ be some smooth strictly convex planar body containing the origin 0 in its interior. The function $d(\cdot,\cdot)$ defined by

$$d(0,z) = \min\{t \ge 0 : z \in tC\},$$

$$d(w,z) = d(0,z-w)$$

is a Finsler metric on $\mathbf{R}^2$ whose balls are precisely the sets positively homothetic to $C$. Thus, using 3.3, 3.2, we obtain an existence and uniqueness result for packing smooth strictly convex sets specified up to homothety. The convex packing theorem of [Sch1] contains the existence part of this statement. The proof there is totally different.

Before we proceed to discuss a geometric application for the packing theorem 3.2, we will state a theorem which gives equivalent conditions for a collection of sets to be packable, but first we need to introduce the concept of compatibility. (See also [Sch2].)

3.4. Definitions. Let $A$ be a subset of the sphere $S^2$, and $p,q \in S^2$. We will say that a curve $\gamma$ *connects* $p$ and $q$ in $A$, if the endpoints of $\gamma$ are $p,q$ and $\mathrm{relint}(\gamma) \subset A$. Here, and in the following, $\mathrm{relint}(\gamma)$ means $\gamma -$ {its endpoints}.

†For the reader who struggles to recall the definition of a Finsler metric, the properties that we will use are that a Finsler metric is a path metric, that its balls are blunt, and that it has the unique extension property. That is, if two length minimizing paths share a nontrivial initial segment, then one is contained in the other.

Note that $p$ and $q$ do not have to be in $A$ to be connected by a curve in $A$, they may be in $\bar{A} - A$.

Let $A, B$ be two closed topological disks in the sphere. We will say that $A$ *cuts* $B$, if there are two points in $B$ – interior($A$) which are not connected by any curve in interior($B$) $- A$. $A$ and $B$ are *incompatible*, if $A \neq B$, and $A$ cuts $B$ or $B$ cuts $A$. Otherwise, they are *compatible*. See Fig. 3.1.

If $A, B$ are disklikes, then they will be considered compatible if $A = B$ or for every connected component $A'$ of $A^c$ and every connected component $B'$ of $B^c$, $A'^c$ and $B'^c$ are unequal and compatible. In other words, either $A = B$, or whenever you adjoin to $A$ all but one of the connected components of its complement and do the same to $B$, the resulting topological disks are unequal and compatible. (This definition is, in turn, compatible with the definition for the case where $A, B$ are topological disks.)

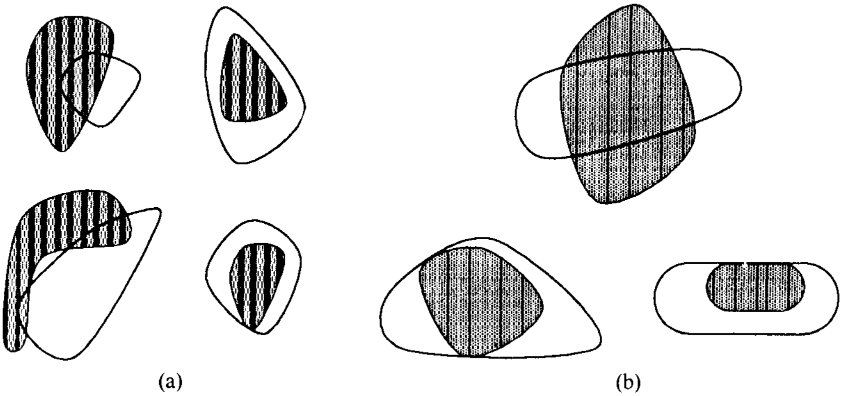
3.5. Theorem. *Let $U$ be a nonempty open simply connected subset of $S^2$, and let $\mathfrak{F}$ be a collection of blunt disklikes in $U$. The following conditions are equivalent.*

(1) *$\mathfrak{F}$ is packable on $U$.*

(2) *$\mathfrak{F}$ is a continuous collection on $U$, it is a 3-manifold, and every two sets $A, B \in \mathfrak{F}$ are compatible.*

(3) *$\mathfrak{F}$ is a continuous collection on $U$, it is a 3-manifold, and to each cornered trilateral $D \subset U$ there is at most one set in $\mathfrak{F}$ which packs it.*

When we say that $\mathfrak{F}$ is a 3-manifold, we mean that it is a 3-manifold in the topology induced by our notion of convergence of subsets of $S^2$. (See the previous section.)



(a) (b)

Fig. 3.1. (a) Some compatible pairs. (b) Some incompatible pairs.

The restriction in Theorem 3.5 that $U$ must be simply connected is not significant: a continuous collection of blunt disklikes $\mathfrak{F}$ is packable on an open set $U$ if, and only if, for every open simply connected $W \subset U$, $\mathfrak{F}|_W = \{A \in \mathfrak{F} : A \subset W\}$ is packable on $W$. This is so, because every trilateral $D \subset U$ is contained in such a $W$.

We now introduce two other classes of collections which are packable, and give a geometric application.

Let $K$ be a smooth strictly convex body in $\mathbf{R}^3$. Consider some closed (affine) half space $H^+$ determined by some plane $H$ which intersects interior$(K)$. The intersection $F(H^+) = H^+ \cap \partial K$ is a smooth topological disk in $\partial K$. Let $\mathfrak{F}$ be the collection of all subsets $F(H^+) \subset \partial K$ obtained in this manner. We may identify $\partial K$ with $S^2$ by some diffeomorphism.

3.6. Theorem. *The above $\mathfrak{F}$ is packable on $\partial K$.*

Proof. We will use 3.5(2). Let $H^+$ be a half space whose boundary, $H$, intersects interior $K$. $H \cap \partial K$ is a (convex) smooth simple closed curve, by strict convexity of $K$, and so $\mathfrak{F}$ is a collection of smooth topological disks. It is clear that $\mathfrak{F}$ is a continuous collection on $\partial K$, and is a 3-manifold.

To prove compatibility, consider two distinct hyperplanes. Their intersection is a line, or empty. A line intersects $\partial K$ in at most two points, by strict convexity of $K$. Thus the boundaries (relative to $\partial K$) of two distinct sets $A, B \in \mathfrak{F}$ intersect in at most two points. If the boundaries intersect in less than two points, then $A, B$ are clearly compatible. Otherwise there is a point $p$ in interior $K$ and in the line of intersection of the two planes. Looking from $p$, $A$ and $B$ look exactly like circles. (This means that the images of their projection on a small sphere centered at $p$ are circles.) Thus they are compatible. It follows from 3.5 that $\mathfrak{F}$ is packable. ■

A dual example is the following. Consider some point $s \in \mathbf{R}^3 - K$, and think of it as a satellite hovering over $K$. Let $F(s)$ be the visual territory of $s$; that is, the set of points in $\partial K$ that the line segment from them to $s$ does not intersect the interior of $K$, and let $F'(s)$ be the closure of the complement of $F(s)$.

We now broaden our horizons a little, and consider $\mathbf{R}^3$ as being contained in real projective 3-space $\mathbf{PR}^3$, then, looking from $s$, there are some directions where $\partial K$ can be seen. If $l$ is a line passing through $s$ and intersecting $K$, then, when looking from $s$ along $l$ in one direction, one sees a point in $F(s)$, and in the other direction, one sees a point in $F'(s)$. We thus conclude that there is a projective description for sets of the type $F(s), F'(s)$: let $o_s$ be one of the two connected components of the set of directions at $s$ such that the oriented line through $s$ in that direction intersects $K$, and let $F(s, o_s)$ be the collection of the first points on $\partial K$

on any of these lines. Let $\mathfrak{F}^*$ be the collection of all sets of the type $F(s, o_s)$, where $s \in \mathbf{PR}^3 - K$, and $o_s$ is as above.

3.7. THEOREM. *The above $\mathfrak{F}^*$ is packable on $\partial K$.*

PROOF. Also easily follows from 3.5. ∎

The following is a geometric application of Theorems 3.6, 3.7.

3.8. COROLLARY. *Let $K$ be a smooth strictly convex body in $\mathbf{R}^3$, and let $P$ be a simple or simplicial 3-polytope. Then there exists a 3-polytope $P'$, which is combinatorically isomorphic to $P$, and which midscribes $K$. That is, all the edges of $P'$ are tangent to $K$.*

In the case where $K$ is a geometric ball, the corollary was proved by Thurston [Th1] (without restrictions on the combinatorial type of $P$), using the Andreev–Thurston theorem, and the proof outlined below is an immediate generalization of that proof, where the Andreev–Thurston theorem is replaced by 3.6, 3.7. This corollary is also true without any restriction on the combinatorial type of $P$, and a proof for that general case (which is not a direct generalization of the proof here or in [Th1]), as well as background and references, appears in [Sch3] (also see [Schu] for background).

I am indebted to Egon Schulte for advising me to try to apply the more general packing theorems to get this corollary.

PROOF OF COROLLARY 3.8, SKETCH. Suppose that $P$ is simple; that is, every vertex of $P$ belongs to precisely three faces. Let $G = G(V)$ be the graph dual to the 1-skeleton of $P$. Since $P$ is simple, $G$ is a triangulation. By 3.6, 3.2, there exists a packing $P = (P_v : v \in V)$ on $\partial K$ whose nerve is $G$, and with $P_v = F(H_v^+)$, $v \in V$, for some half spaces $H_v^+$ (notation described above).

Let $P'$ be the intersection of the opposite half spaces $H_v^-$, and let $G' = G'(V)$ be the adjacency graph of the faces of $P'$. That is, an edge $u \leftrightarrow v$ appears in $G'$, if and only if the intersection of the corresponding planes is an edge of the polyhedron $P'$.

Saying that two sets $P_v, P_u$ touch is the same as saying that the intersection of the corresponding planes $H_u, H_v$ is a line tangent to $K$. (It is tangent at the intersection $P_v \cap P_u$.) Because $P$ is a packing on $\partial K$ with nerve $G$, it follows that every edge of $G$ appears also in $G'$. But $G'$ is planar, and $G$ is a triangulation, so $G = G'$. If $P'$ is bounded, then it is a polytope, and it readily follows that $P'$ is combinatorically isomorphic to $P$. In general, it is not necessarily true that $P'$ is bounded, but it is not difficult to show that one can get such a bounded $P'$.

(Remember, we can choose the sets corresponding to three neighboring vertices in $G$.) An unbounded $P'$ gives a projective polytope midscribing $K$.

For the case where $P$ is simplicial, rather than simple, one can either dualize the result above, using a dual of $K$, or write a dual proof, using 3.7, in place of 3.6. ■

We close this section with a remark about Definition 3.1 and Theorem 3.2. For a continuous collection of blunt disklikes to be packable we have required that for each *cornered* trilateral there is a unique set in the collection which packs that trilateral. Then Theorem 3.2 yields uniqueness and existence of packings in *decent* trilaterals. It is possible to weaken the hypotheses and to require that for each decent trilateral there is a unique set in the collection which packs it, and still get the same conclusions. However, the modifications needed in the proof are technical and uninteresting.

## 4. Proof of the Metric Packing Theorem 3.3

PROOF OF 3.3. We will use Theorem 3.5, which will be proved in later sections.

Let $\mathfrak{F}_d$ be the collection of proper (left) $d$-balls. It is immediate that $\mathfrak{F}_d$ is a continuous collection of blunt disklikes on $U$ and, clearly, it is a 3-manifold, parametrized by center and radius. It follows from the remark following 3.5 that it is sufficient to consider the case where $U$ is simply connected, and we restrict our attention to this situation. Let $D = (D_1, D_2, D_3)$ be a cornered trilateral in $U$. By 3.5(3), it remains to prove that there is at most one (left) $d$-ball which packs $D$.

Suppose that $p$ and $q$ are centers of $d$-balls which pack $D$. We must show that $p = q$. Assume $p \neq q$. Let $r = d(p, \partial D)$ be the radius of the ball $B$ with center $p$ which packs $D$. We know that $B$ doesn't intersect the vertices of $D$, because $B$ is blunt and $D$ is cornered.

Let $\gamma_1$, $\gamma_2$ and $\gamma_3$ be paths of length $r$ from $p$ to $D_1$, $D_2$ and $D_3$, respectively. Consider some point $s \in \gamma_1$, $s \neq p$. We will now show that the distance from $s$ to $D_2 \cup D_3$ is bigger than its distance to $D_1$. If $d(s, D_2 \cup D_3) \leq d(s, D_1)$, then the path $\gamma$ which follows $\gamma_1$ from $p$ to $s$, and then takes a shortest route to $D_2 \cup D_3$ would have length

$$\text{length}(\gamma) = d(p,s) + d(s, D_2 \cup D_3) \leq d(p,s) + d(s, D_1) = \text{length}(\gamma_1) = r.$$

This means that it is a shortest path from $p$ to $D_2 \cup D_3$, and has length $r$. Because the metric has the unique extension property (two length minimizing paths of the same length which share a nontrivial initial segment must be the same), it follows that $\gamma = \gamma_1$. This implies that the terminal point of $\gamma_1$ is on $D_1 \cap (D_2 \cup D_3)$.

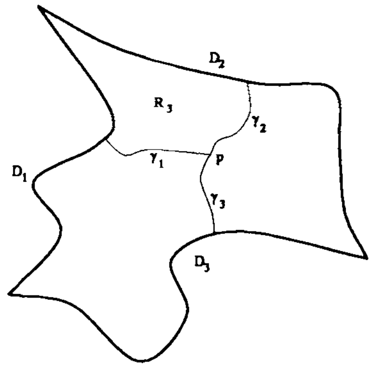



Fig. 4.1. The paths $\gamma_1, \gamma_2, \gamma_3$ and the region $R_3$.

However, this is impossible, because $B$ does not intersect the vertices of $D$. We thus conclude that $d(s, D_2 \cup D_3) > d(s, D_1)$. A similar result holds for points $s$ in $\gamma_2 - \{p\}$, or in $\gamma_3 - \{p\}$. It also follows that the paths $\gamma_1, \gamma_2, \gamma_3$ are disjoint, except at $p$.

Let $R_3$ be the closure of the region of $D$ which is determined by $\gamma_1$ and $\gamma_2$ and is disjoint from $D_3$. See Fig. 4.1. By symmetry we may, and will, assume that $q \in R_3$. Let $\delta_3$ be a shortest path from $q$ to $D_3$. It must intersect $\gamma_1 \cup \gamma_2$. Suppose that $s$ is a point in this intersection.

Consider the case where $s$ is on $\gamma_1$, and is not $p$. We have seen above that $d(s, D_1) < d(s, D_2 \cup D_3)$. But, because $s$ is on $\delta_3$, a similar argument, with $q$ replacing $p$, shows that $d(s, D_3) < d(s, D_1 \cup D_2)$, if $s \neq q$, or $d(s, D_3) = d(s, D_1 \cup D_2)$, if $s = q$. So, in both cases, we get a contradiction. A contradiction is reached similarly, if $p \neq s \in \gamma_2$. The remaining case, $p = s$, would give $d(q, \partial D) = d(q, D_3) > d(p, \partial D)$. A symmetric argument (exchanging $p, q$) would lead to the opposite conclusion, and thus a contradiction is reached in this last case also. This completes the proof. ■

## 5. Uniqueness of packings

The major tool at our disposal is the incompatibility theorem from [Sch2], which is restated here for the convenience of the reader.

5.1. Incompatibility Theorem ([Sch2]). *Let $T = T(V)$ be an oriented triangulation of a quadrilateral with boundary vertices $a, b, c, d$, in clockwise order with respect to the other vertices of $T$ (if such exist). Let $D = (D_1, D_2, D_3, D_4)$ be a* *quadrilateral in* $S^2$. *Suppose that* $Q = (Q_v : v \in V)$ *and* $P = (P_v : v \in V)$ *are two nondegenerate packings in D, both having nerve T, and their orientations agree with that of T. Further suppose that* $Q_v$ *and* $P_v$ *are disklike for* $v \in V - \{a,b,c,d\}$, *that* $P_a \subset D_1$, $Q_b \subset D_2$, $P_c \subset D_3$, $Q_d \supset D_4$, *and that* $P_v$ *is disjoint from* $Q_d$ *for* $v \in V - \{a,c,d\}$. *Then there is some vertex* $v \in V - \{a,b,c,d\}$ *for which* $Q_v$ *and* $P_v$ *are incompatible.*

Our intermediate goal is to prove

5.2. Proposition. *Theorem* 3.2 *holds if the collections* $\mathfrak{F}_v$ *also satisfy the condition that any two sets* $A, B \in \mathfrak{F}_v$ *are compatible.*

But first we need

5.3. Uniqueness Lemma. *Let* $T = T(V)$ *be a triangulation of* $S^2$, *and let* $[a,b,c]$ *be a distinguished triangle in T. Let* $D = (D_a, D_b, D_c) \subset S^2$ *be some trilateral. Let* $Q = (Q_v : v \in V)$ *and* $P = (P_v : v \in V)$ *be two nondegenerate packings in D having nerve T and satisfying* $Q_v = D_v = P_v$, $v = a,b,c$. *Further assume that, for* $v \in V - \{a,b,c\}$, $Q_v$ *and* $P_v$ *are disklike compatible sets. Then these two packings are the same*: $Q_v = P_v$, $v \in V$.

Proof of 5.3. The proof will be an easy application of the incompatibility theorem. Consider first the case where all the $Q_v, P_v$, $v \neq a,b,c$, are topological disks. We proceed by induction on the number of vertices in $T$. The case $V = \{a,b,c\}$ is clear. So we will assume that $T$ has more than three vertices, and that the lemma holds for triangulations with fewer vertices than $T$.

Let $d$ be the unique vertex other than $b$ which forms a triangle together with $a$ and $c$. In other words, $[a,c,d]$ and $[a,b,c]$ are the two triangles that have the edge $a \leftrightarrow c$ on their boundary.

With the intention of arriving at a contradiction, we will assume now that $Q_d \neq P_d$. Let $r$ be a point in the intersection $Q_d \cap D_a$, and let $s$ be a point in the intersection $Q_d \cap D_c$. Likewise, let $r' \in P_d \cap D_a$, $s' \in P_d \cap D_c$. Because the packings are nondegenerate, $r \neq s$ and $r' \neq s'$. $r$ and $s$ are certainly in the complement of interior$(P_d)$, and therefore, by compatibility of $Q_d$ and $P_d$, there is a simple curve $\alpha$ which connects them in interior$(Q_d - P_d)$. Likewise, there is a simple curve $\alpha'$ which connects $r', s'$ in interior$(P_d - Q_d)$. There are two cases to consider. Either $\alpha$ separates $P_d$ from $D_b$ in $D$, or $\alpha'$ separates $Q_d$ from $D_b$ in $D$. See Fig. 5.1. (Since $\alpha$ can possibly intersect $P_d$ at $r$ or $s$, what we mean by '$\alpha$ separates $P_d$ from $D_b$ in $D$' is that $P_d - \{r,s\}$ and $D_b$ are in different connected components of $D - \alpha$. Similarly for the other case.) By symmetry, we need to consider only one of these cases, so assume that $\alpha'$ separates $Q_d$ from $D_b$ in $D$.

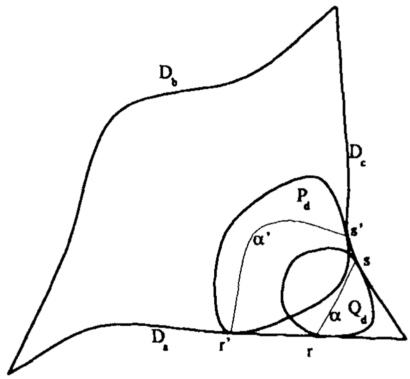



Fig. 5.1. $\alpha'$ separates $Q_d$ from $D_b$.

From this fact, that $\alpha'$ separates $Q_d$ from $D_b$ in $D$, it follows that all the sets $P_v$, $v \neq d$, are disjoint from $Q_d$, because they must be on the side of $\alpha'$ which contains $D_b$ (since $T - \{a,c,d\}$ is edge-connected).

The curve $\alpha$ cuts $D$ into a trilateral and a quadrilateral. Call the quadrilateral $\hat{D}$. $\hat{D} = (\partial\hat{D} \cap D_a, \partial\hat{D} \cap D_b, \partial\hat{D} \cap D_c, \alpha)$. Define $\hat{Q}_v = Q_v \cap \hat{D}$ and $\hat{P}_v = P_v \cap \hat{D}$ for all $v \in V$. Then $\hat{Q}_v = Q_v$ for $v \neq a,c,d$ and $\hat{P}_v = P_v$ for $v \neq a,c$. The packings $\hat{Q}$ and $\hat{\mathrm{P}}$ in the quadrilateral $\hat{D}$ satisfy the hypotheses of the quadrilateral theorem, and therefore there is some $v \in V - \{a,b,c,d\}$ for which $\hat{Q}_v$ and $\hat{P}_v$ are incompatible. But this would imply that $Q_v$ and $P_v$ are incompatible, contrary to our assumptions. This contradiction shows that $Q_d = P_d$.

Having established $Q_d = P_d$, the equivalence $Q_v = P_v$, $v \in V$, follows easily from the inductive hypothesis. This completes the proof of the uniqueness lemma in the case where the $Q_v, P_v$, $v \neq a,b,c$, are topological disks.

For $v \neq a,b,c$ let $Q_v'$ be the topological disk obtained from $Q_v$ by adjoining to $Q_v$ the connected components of $Q_v^c$ (the complement of $Q_v$) except for the connected component whose closure contains all the other sets in the packing $Q$. Let $Q'$ be the packing $(Q_v' : v \in V)$, with $Q_v' = Q_v$, $v = a,b,c$, and let the packing $P' = (P_v' : v \in V)$ be defined analogously. The above proof implies that $P' = Q'$. From compatibility of $P_v$ and $Q_v$, $v \neq a,b,c$, it follows that $P_v = Q_v$, completing the proof. ■

## 6. Existence of packings

Proof of 5.2. The uniqueness part clearly follows from the uniqueness lemma, and thus it remains to prove existence. Existence will also be proved by induction. The basic approach is to use uniqueness to show that the packings for triangulations simpler than $T$ are continuous in the data, in particular, continuous in the given trilateral. From this continuity, the existence for the triangulation $T$ then follows.

The base of the induction is the case where $a, b, c$ are the only vertices in $V$, and there is nothing to prove in that case. Thus we assume that $T$ has more than three vertices, and that the theorem has been proven for triangulations with fewer vertices than $T$ has.

We will deal first with the easy case, in which there are three vertices of $T$, say $f, g, h$, of which every two share an edge, but they do not form a triangle in $T$. In this case we say that $T$ is *decomposable* for the following reasons. The three edges going between these vertices $f, g, h$ determine two regions of $T$, say $R_1, R_2$. Let $T_1$ be the triangulation formed from $T$ by dropping all the vertices in $R_1$ and all the edges adjacent to them. Similarly, let $T_2$ be the triangulation formed from $T$ by dropping all the vertices in $R_2$ and all the edges adjacent to them. (The three vertices $f, g, h$ form triangles in $T_1$ and $T_2$. That is the reason why $T_1$ and $T_2$ are triangulations.) Because $f, g, h$ do not form a triangle in $T$, we know that there are some vertices in each of the regions $R_1, R_2$. Therefore the triangulations $T_1, T_2$ have fewer vertices than $T$, and the inductive hypothesis can be applied to them. Without loss of generality, assume that the three vertices $a, b, c$ are in $T_1$.

From the inductive hypothesis applied to $T_1$ we get a packing $Q^1 = (Q_v^1 : v \in V_1)$ in $D$ whose nerve is $T_1$ and which satisfies $Q_v^1 = D_v$, $v = a, b, c$, and $Q_v^1 \in \mathfrak{F}_v$, $v \in V_1 - \{a, b, c\}$. The three touching sets $Q_v^1$, $v = f, g, h$, determine a decent trilateral $D^2$ which is disjoint from all the other sets $Q_v^1$, $v \in V_1 - \{f, g, h\}$. We now apply the inductive hypothesis to $T_2$, to obtain a packing $Q^2 = (Q_v^2 : v \in V_2)$ whose nerve is $T_2$ and which satisfies $Q_v^2 = Q_v^1$, $v = f, g, h$, and $Q_v^2 \in \mathfrak{F}_v$, $v \in V_2 - \{f, g, h\}$, $Q_v^2 \subset D_2$, $v \in V_2 - \{f, g, h\}$. Pasting the two packings $Q^1$ and $Q^2$ together gives our desired packing.

Having completed the inductive step for the case in which $T$ is decomposable, we will assume from now on that it is not decomposable. As in the proof of uniqueness, let $d$ be the unique vertex other than $b$ which forms a triangle together with $a$ and $c$. If $a, b, c, d$ are the only vertices of $T$, then existence follows from the hypothesis on $\mathfrak{F}_d$. Therefore we assume that this is not the case. Let $p_0$ be the point of intersection of $D_a$ and $D_c$.

6.1. Lemma. *In $\mathfrak{F}_d$ there is a continuous one parameter family of sets $P_d^t$, $t \in (0,1]$, which satisfies*:

(1) $P_d^t \subset D$, $t \in (0,1]$,

(2) $P_d^t$ *touches each of the two arcs* $D_a, D_c$ *for* $t \in (0,1]$,

(3) *$P_d^1$ touches $D_b$, and*

(4) *as $t \to 0$ the sets $P_d^t$ shrink to $p_0$.*

PROOF OF LEMMA 6.1. Take a continuous one parameter family of simple curves $\gamma^t$, $t \in (0,1]$, in $D - \{p_0\}$ so that (a) $\gamma^1 = D_b$, (b) each $\gamma^t$ has one endpoint on $D_a$ and the other endpoint on $D_c$, (c) the trilateral $E^t \subset D$ bounded by parts of $\gamma^t, D_a, D_c$ is decent, and (d) $E^t$ is contained in an arbitrarily small neighborhood of $p_0$ for $t$ sufficiently small. Clearly such a family $(\gamma^t)$ exists.

For each $t \in (0,1]$, $\mathfrak{F}_d$ contains a unique set $P_d^t \subset E^t$ which touches all three edges of $E^t$. To verify that $P_d^t$ is continuous in $t$, consider some point $t \in (0,1]$ and a sequence $t_i \in (0,1]$ converging to $t$. Since $\mathfrak{F}_d$ is continuous, by passing to a subsequence if necessary, we may, and will, assume that $P_d^{t_i}$ converges to a set $A$, and $A \in \mathfrak{F}_d$ or $A$ consists of a single point. But clearly, $A$ is contained in the trilateral $E^t$ and touches each of its edges. So $A$ is not a single point, $A \in \mathfrak{F}_d$, and we have $A = P_d^t$, by uniqueness. This establishes continuity. The other claims of the lemma are obvious. ■

CONTINUATION OF THE PROOF OF 5.2. Let $P_d^t$, $t \in (0,1]$, be as in the lemma. Fix some $t \in (0,1]$. In $D - P_d^t$ there is a unique connected component whose boundary intersects each one of the sets $P_d^t, D_b, D_c$. See Fig. 6.1. Let us denote the closure of this connected component by $D^t$. We will view $D^t$ as a (decent) trilateral $D^t = (D_a^t, D_b^t, D_c^t)$, with $D_a^t = D^t \cap (P_d^t \cup D_a)$, $D_b^t = D^t \cap D_b$ and $D_c^t = D^t \cap D_c$. It should be emphasized that we are joining $D^t \cap P_d^t$ and $D^t \cap D_a$ to one edge on $D^t$.

Let $T'$ be the graph obtained from $T$ by dropping the vertex $d$ and all the edges

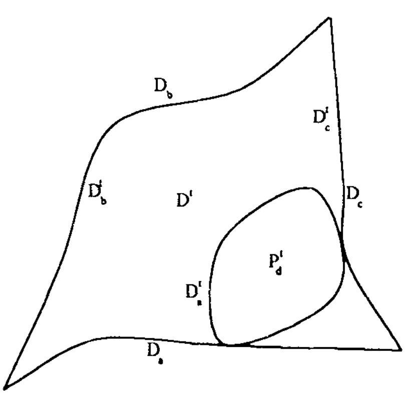



Fig. 6.1. The trilateral $D^t$.

adjacent to it, and inserting an edge between $a$ and any neighbor of $d$ which isn't already a neighbor of $a$. From our assumption that $T$ is not decomposable, it follows that $T'$ is a triangulation. See Fig. 6.2. $T'$ is (isomorphic to) the triangulation obtained by coalescing $a$ and $d$ in $T$.

We now use the inductive hypothesis for the triangulation $T'$ with respect to the trilateral $D^t$. It implies that there exists a packing $Q^t = (Q_v^t : v \in V - \{d\})$ in $D^t$ whose nerve is $T'$, and which satisfies $Q_v^t = D_v^t$, $v = a,b,c$, and $Q_v^t \in \mathfrak{F}_v$, $v \in V - \{a,b,c,d\}$. Let $P^t = (P_v^t : v \in V)$ be the packing defined by $P_v^t = Q_v^t$, $v \in V - \{a,b,c,d\}$, $P_v^t = D_v$, $v = a,b,c$, and $P_d^t$ as defined above. The packing $P^t$ is almost what we need: for every $v, w \in V$ which neighbor in $T$, we know that the corresponding sets $P_v^t, P_w^t$ touch, except for the case $v \in \{a,d\}$, $w \in V - \{a,b,c,d\}$ and the symmetric case $w \in \{a,d\}$, $v \in V - \{a,b,c,d\}$. When $w$ neighbors with $a$ or $d$, we know that $P_w^t$ touches $P_a^t$ or $P_d^t$.

Now let $e$ be the vertex other than $c$ which neighbors with $a$ and with $d$ in $T$, as in Fig. 6.2. The case $e = b$ is easily ruled out by our assumptions that $T$ is not decomposable and that $a,b,c,d$ are not the only vertices of $T$. Proposition 5.2 now follows from the following two lemmas.

6.2. Lemma. *There is some $t \in (0,1]$ for which $P_e^t$ touches both $P_a^t$ and $P_d^t$.*

6.3. Lemma. *If $P_e^t$ intersects both $P_a^t$ and $P_d^t$, then the packing $P^t$ has nerve $T$.*

Proof of Lemma 6.3. By the above, all that needs to be verified is that for $v \in \{a,d\}$, $w \in V - \{a,b,c,d,e\}$ which neighbor in $T$, $P_v^t$ and $P_w^t$ intersect. (The set of edges in the nerve cannot strictly contain the set of edges in a triangulation,

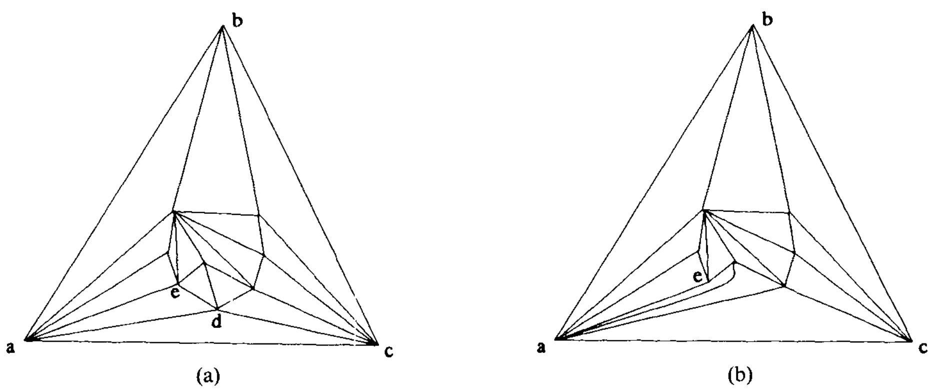



Fig. 6.2. (a) The triangulation $T$. (b) The resulting $T'$.

by planarity.) We will consider only the situation $v = d$. The other case is treated similarly.

Let $b = v_0, v_1, \ldots, v_n = c$ be the neighbors of $a$ in $T$ in circular order. We have $e = v_{n-2}$, $d = v_{n-1}$. Because $v_0 \to v_1 \to \cdots \to v_{n-2} = e$ is a path in $T'$, and because $P_e^t$ touches $P_d^t$ which touches $D_c$, there is a path $\alpha$ from $D_b$ to $D_c$ with $\alpha \subset \bigcup_{j=1}^{n-1} P_{v_j}^t$. It can also be arranged that $\alpha$ is disjoint from every $P_v^t$, $v \neq v_0, v_1, \ldots, v_n$. $D - \alpha$ consists of two connected components. One of them intersects $D_c^t$ and the other intersects $D_a$. The union $\bigcup_{v \in V - \{a, v_0, \ldots, v_n\}} P_v^t$ is connected (by indecomposability), and intersects $D_c^t$. Therefore it doesn't intersect $D_a$. Clearly, $w \notin a, v_0, \ldots, v_n$ (again by indecomposability), and so $P_w^t$ is disjoint from $D_a$. But $w$ neighbors with $a$ in $T'$. Therefore $P_w^t$ intersects $Q_a^t = D^t \cap (P_d^t \cup D_a)$. Thus $P_w^t$ intersects $P_d^t$, and the proof of 6.3 is complete. ■

Proof of Lemma 6.2. Let $A \subset (0,1]$ be the set of parameters $t$ for which $P_e^t$ intersects $P_a^t$, and let $B \subset (0,1]$ be the set of parameters $t$ for which $P_e^t$ intersects $P_d^t$. By construction, $P_e^t$ always intersects $P_a^t \cup P_d^t$, and thus $A \cup B = (0,1]$. We need to show that $A$ and $B$ intersect.

In order to see that $A$ and $B$ are closed sets, we will first show that the packings $P^t$ depend continuously on $t$. The argument is like the argument in the proof of Lemma 6.1 showing that $P_d^t$ depends continuously on $t$. Let $t_n$, $n = 1,2,\ldots$, be a sequence in (0,1] converging to $t \in (0,1]$. Let $K_v = \lim_{n\to\infty} P_v^{t_n}$, $v \in V$, assuming, as we may, by continuity of the $\mathfrak{F}_v$, that these limits exist. (We use the definition of set convergence introduced in Section 2.) $K = (K_v : v \in V - \{d\})$ is an (*a priori* possibly degenerate) packing. It is clear that $K_v$ and $K_w$ intersect whenever $v \leftrightarrow w$ is an edge in $T'$. Every $K_v$, $v \in V - \{a,b,c\}$, which is not a single point must be in $\mathfrak{F}_v$, by continuity of the $\mathfrak{F}_v$, and therefore is a blunt set. We want to see that all $K_v$'s are not points. To see this, let $U$ be a connected component of the set of vertices in $V$ which correspond to $K_v$'s which are points. We have $a, b, c \notin U$. Let $U'$ be the set of vertices in $V - U$ which neighbor with some vertex of $U$. Clearly, $U'$ contains at least three vertices. All the sets $K_v$, $v \in U$, intersect, and therefore they are the same point. This shows that the intersection of all the sets $K_u$, $u \in U'$, is nonempty, but this is impossible, because it is impossible for more than two blunt sets in a packing to touch at a single point. This contradiction leads us to conclude that none of the $K_v$'s are single points.‡

So $K$ is a packing, and the nerve of it contains every edge in $T'$. Being a planar

‡The reader may wish to compare this argument with the ring lemma of [R-S]. They are closely related.

graph, this forces the nerve to be $T'$. From uniqueness we conclude that $K = P^t$, and continuity is established. This continuity implies that $A$ and $B$ are closed.

To see that $B \neq (0,1]$ consider a packing $P^0 = (P_v^0 : v \in V - \{d\})$ which satisfies the conditions $P_v^0 = D_v$, $v = a,b,c$, $P_v^0 \in \mathfrak{F}_v$, $v \in V - \{a,b,c,d\}$, and has nerve $T'$. This packing exists, by the inductive hypothesis. The sets $P_v^0$, $v \neq a,c$ cannot contain the point $p_0$, where $D_a$ and $D_c$ meet. Therefore they are all disjoint from $P_d^t$, for $t$ sufficiently small. Thus, by uniqueness, $P_v^t = P_v^0$, $v \neq a,c,d$, for $t$ sufficiently small. For these $t$ we then have $P_e^t$ disjoint from $P_d^t$. So $B \neq (0,1]$.

For $t = 1$ the set $P_d^1$ intersects with $D_b$. This shows that $D^1$ is disjoint from $D_a$. Thus $P_e^1$ cannot intersect $D_a$, and $A \neq (0,1]$.

We have seen that $A$ and $B$ are closed, that their union is $(0,1]$, and that $A \neq (0,1] \neq B$. Because $(0,1]$ is connected, it follows that $A$ and $B$ must intersect. This establishes Lemma 6.2, and also Proposition 5.2. ■

## 7. Proof of the Main Packing Theorem 3.2

In the proof of 3.2 we may restrict our attention to some neighborhood of $D$. It is therefore sufficient to consider only the case where $U$ is simply connected. By appeal to Proposition 5.2, we see that 3.2 reduces to the following lemma.

7.1. Compatibility Lemma. *Let $U \subset S^2$ be open and simply connected. Let $\mathfrak{F}$ be a packable collection on $U$. Then any two sets in $\mathfrak{F}$ are compatible.*

Proof. Let $E,F$ be two sets in $\mathfrak{F}$. Let $E'$ be the set obtained from $E$ by adjoining to $E$ all but one of the connected components of the complement of $E$, and let $F'$ be similarly related to $F$.

If $E' = F'$, then, clearly, there is some cornered trilateral which they both pack. Therefore $E$ and $F$ both pack it, and so it follows that $E = F$. With the intention of reaching a contradiction we assume that $E,F$ are not compatible. From the above, it follows that $E' \neq F'$ are incompatible. Say $E'$ cuts $F'$.

*Case 1, $E' \supset F'$.* Let $p,q$ be two distinct points in $F' -$ interior$(E')$ which cannot be connected by a path in interior$(F' - E')$. Because $E' \supset F'$, $p,q \in E'$, and therefore $p,q \in \partial E'$. Let $\alpha$ be a simple path connecting $p$ and $q$ in interior$(F')$. It obviously cuts $E'$ in two, and we see that also $F'$ cuts $E'$. If $F' \supset E'$, then $F' = E'$, contrary to our previous conclusions. Thus, by exchanging $F$ and $E$, we have reduced the proof to the second case:

*Case 2, $F' - E' \neq \emptyset$.* Consider some point $r \in F' - E'$. Because $p$ and $q$ cannot be connected by a curve in interior$(F' - E')$, it follows that the same is true for $p$ and $r$ or for $q$ and $r$. Thus we may, and will, assume that $p \notin E'$. There is some curve connecting $p$ and $q$ in the complement of $E'$. By replacing $p$ and/or $q$ by other points along this curve, we can make a reduction to the situation where $p$ and $q$ are connected by some curve in the complement of $E' \cup F'$. Assume that this is the case.

Let $\alpha_1$ and $\alpha_2$ be the two arcs of $\partial F'$ between $p$ and $q$. If $E'$ is disjoint from the relative interior of $\alpha_1$ then one can perturb $\alpha_1$ slightly so that it connects $p$ and $q$ in interior$(F' - E')$. This contradicts our assumption, and therefore there is a point $s \in \text{relint}(\alpha_1) \cap E'$. Similarly, let $t$ be a point in relint$(\alpha_2) \cap E'$. Summarizing, we have points $p, s, q, t$ in cyclic order on $\partial F'$ with $s, t \in E'$, $p \notin E'$, and $p, q$ are connected by a curve in the complement of $E' \cup F'$.

*Case 2a,* $E' - (\{s,t\} \cup \text{interior}(F')) \neq \emptyset$. In this case we construct a cornered trilateral $D$ so that both the sets $E'$ and $F'$ pack it. See Fig. 7.1a.

To make the construction explicit, let $W$ be the connected component of the complement of $E' \cup F'$ which has $p$ and $q$ on its boundary. We start with a simple closed curve $\gamma$ in $W$ which follows closely $\partial W \cap (E' \cup F')$, and goes around $E' \cup F'$. Near $p$ we fix a point $v_0 \in \gamma$ to be a vertex of the trilateral. We modify $\gamma$ slightly near $v_0$, so that it has a corner at $v_0$, and so that it touches $F'$ right before and right after $v_0$. We also slide $\gamma$ to $\partial(E' \cup F')$ so that it touches $q$ and touches $E'$ in at least three points, but remains in $\overline{W}$. It is possible to do this because we are assuming $E' - (\{s,t\} \cup \text{interior}(F')) \neq \emptyset$. Furthermore, we make sure that of the points where $\gamma$ touches $E'$ there will be at least one in each relatively open arc of $\gamma$ from $v_0$ to $q$. (Note that $s, t \in E'$.) Now choose points $v_1, v_2$ on $\gamma$ that are just slightly after the first intersection of $\gamma$ with $E'$, when traveling from $v_0$ in each of the two directions. Make small corners in $\gamma$ on $v_1$ and $v_2$, by

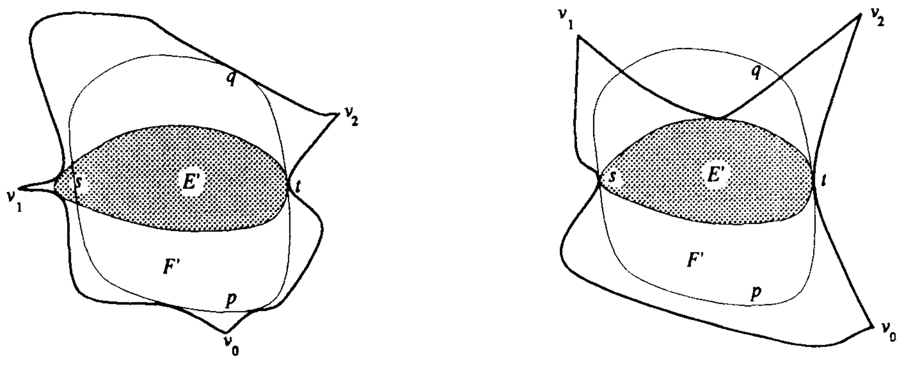



Fig. 7.1. The constructed trilaterals.

first moving slightly away from $E' \cup F'$, if necessary. This gives a cornered trilateral which both $F'$ and $E'$ pack. Therefore both $E$ and $F$ pack this cornered trilateral, contrary to our assumptions.

*Case 2b,* $E' \subset (\{s,t\} \cup \operatorname{interior}(F'))$. We have $\partial F' \cap E' = \{s,t\}$. We construct another cornered trilateral. See Fig. 7.1b. Take a simple closed curve $\gamma$ which touches $F'$ at $s$ and $t$, is otherwise disjoint from $F'$ in some neighborhoods of $s$ and $t$, touches $E'$ in some other point, say in $w$, but is disjoint from $E' - \{s,t,w\}$. ($\gamma$ can be obtained from $\partial F'$ by some simple modifications.) Choose three vertices, and make three small corners in $\gamma$ interleaved between the points $s,t,w$, and call the resulting cornered trilateral $D$. $E'$ is packed in $D$, and therefore $E$ is also packed in $D$.

Now perturb this trilateral $D$ in the vicinity of $s$, so that the boundary of the new trilateral $\hat{D}$ is disjoint from $F'$ near $s$. There is some set $\hat{E} \in \mathfrak{F}$ which packs $\hat{D}$. By continuity and uniqueness, $\hat{E}$ is as close as we wish to $E$, provided $\hat{D}$ is close enough to $D$. Therefore, provided the perturbation was small, $\hat{E}$ intersects the boundary of $F'$ near $s$ and $t$ and is disjoint from $p$ and $q$. Furthermore, $\hat{E}$ is not contained in $F'$ in the vicinity of $s$. This reduces the proof to Case 2a.

Thus the proofs of the lemma and of 3.2 are complete. ■

7.2. Remark. Only in 2b was continuity used. The proof for the other cases shows that if two topological closed disks are not compatible, then either there is some trilateral which both of them pack, or one is contained in the union of the interior of the other and two points on its boundary.

## 8. Equivalent definitions for 'packable': Proof of 3.5

Proof of (1) ⇒ (2). It follows from Lemma 7.1 that every two sets in $\mathfrak{F}$ are compatible. To see that $\mathfrak{F}$ is a 3-manifold, consider first the case that $U$ is an open disk. By applying a diffeomorphism, we will assume that $U = \mathbf{R}^2$. Choose a (geometric) triangle $T$ in $\mathbf{R}^2$. To each homothetic copy of $T$ there exists a unique set in $\mathfrak{F}$ which packs that triangle. This gives a bijective mapping between $\mathfrak{F}$ and the set of triangles homothetic to $T$. It is easy to see that this mapping is bi-continuous, and $\mathfrak{F}$ is topologically $\mathbf{R}^3$.

If $U$ is not a topological open disk, then $U = S^2$. In this case, for $A \in \mathfrak{F}$ take some $p \notin A$. The collection $\{B \in \mathfrak{F} : p \notin B\}$ is a neighborhood of $A$ in $\mathfrak{F}$ which is homeomorphic to $\mathbf{R}^3$. So $\mathfrak{F}$ is a 3-manifold.

Proof of (2) ⇒ (3). This is easy, and also follows from the Uniqueness Lemma 5.3.

PROOF OF (3) ⇒ (1). We need to see that to each cornered trilateral $D \subset U$ there is some $A \in \mathfrak{F}$ which packs it. Again, consider first the case that $U$ is topologically an open disk. Because we can use approximations and convergence, it is sufficient to examine the case where $D$ is a diffeomorphic image of a (geometric) triangle. By applying a diffeomorphism, assume that $U = \mathbf{R}^2$ and $D$ is a triangle. To each compact $K \subset \mathbf{R}^2$ containing more than one point, there is a unique homothetic copy $f(K)$ of $D$ so that $K$ packs $f(K)$. Let $f'$ be the restriction of $f$ to $\mathfrak{F}$.

We need to show that $D \in f'(\mathfrak{F})$. The mapping $f'$ is, obviously, continuous, and is injective, by (3). From our definition of 'continuous collection', it follows that $f'$ is a proper map. Because $f'$ is injective and $\mathfrak{F}$ is a 3-manifold, using invariance of domain, we see that $f'$ is an open map from $\mathfrak{F}$ to the collection of triangles homothetic to $D$. Being open and proper, it follows that $f'$ is surjective. (The set of triangles homothetic to $D$ is connected.) This proves (3) ⇒ (1) for the case $U \neq S^2$.

If $U = S^2$, take some $A \in \mathfrak{F}$ and some $p \notin A$. $\{B \in \mathfrak{F} : p \notin B\}$ is then packable on $U - \{p\}$. It follows that to each $q \in S^2$ there is some $A \in \mathfrak{F}$ so that $q \notin A$. Choose $q \notin D$. $\{B \in \mathfrak{F} : q \notin B\}$ is packable on $U - \{q\}$, and so there is some $B \in \mathfrak{F}$ which packs $D$. ■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